

# Coloring Hanoi and Sierpiński graphs<sup>☆</sup>

Andreas M. Hinz<sup>a,b,\*</sup>, Daniele Parisse<sup>c</sup>

<sup>a</sup> Department of Mathematics, LMU München, Theresienstraße 39, 80333 Munich, Germany

<sup>b</sup> Fakultät für Mathematik und Informatik, FernUniversität in Hagen, 58084 Hagen, Germany

<sup>c</sup> EADS Deutschland GmbH, 81663 Munich, Germany

## ARTICLE INFO

### Article history:

Available online 15 September 2011

### Keywords:

Hanoi graphs  
Sierpiński graphs  
Vertex-coloring  
Edge-coloring  
Total coloring  
Planarity

## ABSTRACT

It is shown that all Hanoi and Sierpiński graphs are in edge- and total coloring class 1, except those isomorphic to a complete graph of odd or even order, respectively. New proofs for their classification with respect to planarity are also given.

© 2011 Elsevier B.V. All rights reserved.

## 0. Introduction

We will provide a comprehensive overview of vertex-, edge-, and total chromatic numbers for the two-parameter families of *Hanoi graphs*  $H_p^n$  and *Sierpiński graphs*  $S_p^n$  with base  $p \in \mathbb{N}$ ,  $p \geq 3$ , and exponent  $n \in \mathbb{N}_0$ . They arose from the mathematical models of the famous *Tower of Hanoi* (TH) problem (cf. [23,8,9,11]) and its variant, the *Switching Tower of Hanoi* (STH) of [17], and will be defined in Sections 0.1 and 0.2, respectively. The trivial case  $n = 0$  will only be used for this definition, while for  $n = 1$  it will turn out that both  $H_p^1$  and  $S_p^1$  are isomorphic to the complete graph  $K_p$ , and their respective coloring properties will be summarized in Section 1. To avoid unpleasant case distinctions, we will thereafter, in Sections 2–4 on vertex-, edge-, and total colorings, respectively, restrict ourselves to values of  $n$  strictly greater than 1.

Vertex coloring (Section 2) can easily be done for both families of graphs simultaneously, leading to Theorem 2. However, the different behavior of complete graphs with respect to edge- and total colorings depending on the parity of their order  $p$  (see Theorem 1) will necessitate a thorough distinction of odd and even  $p$  in later sections. For instance, in the Sierpiński case we can explicitly construct optimal edge colorings for even  $p$  (Lemma 4) and optimal total colorings for odd  $p$  (Lemma 4'), whereas we need somewhat more involved induction proofs in the antithetical cases (Lemmas 5 and 5'). For Hanoi graphs the procedure is even more sophisticated. The general idea is to start off from  $H_3^n$  (Lemma 1, where we actually show a little more, and Lemma 1') and  $H_p^1$  and then step from  $H_{p-1}^{n-1}$  to  $H_p^n$  (Lemmas 3 and 3'). But again, since  $H_p^1 \cong K_p$  does not behave uniformly with respect to  $p$ , we have to take extra care of  $H_p^2$  for even  $p$  in Lemmas 2 and 2'. Moreover, for the total case and odd  $p > 3$  we have to deviate from our general strategy and use the results already obtained for the edge-colorings.

The vertex- and total chromatic numbers for all  $H_p^n$  and  $S_p^n$ ,  $n \geq 2$ , can then be found in Theorems 3 and 4, which follow easily from the lemmas.

<sup>☆</sup> ©A.M. Hinz, D. Parisse 2011.

\* Corresponding author.

E-mail addresses: [hinz@math.lmu.de](mailto:hinz@math.lmu.de) (A.M. Hinz), [daniele.parisse@eads.com](mailto:daniele.parisse@eads.com) (D. Parisse).

### 0.1. Hanoi graphs

The vertex set of Hanoi graph  $H_p^n$  is the set  $V(H_p^n) = [p]_0^n$  of regular states of the TH puzzle, i.e. the legal distributions of  $n$  discs among  $p$  pegs, which we will denote by  $s = s_n \dots s_1$ , where  $s_d$  is the label of the peg disc  $d$  is lying on. (For  $q \in \mathbb{N}_0$ , we use the notation  $[q]_0 := \{0, \dots, q-1\}$ ,  $[q] := \{1, \dots, q\}$  for the initial segments of length  $q$  in  $\mathbb{N}_0$  and  $\mathbb{N}$ , respectively.) The edge sets are defined according to the rules of the TH by

$$E(H_p^n) = \{ \{ \underline{s}i\bar{s}, \underline{s}j\bar{s} \}; d \in [n], \underline{s} \in [p]_0^{n-d}, i, j \in [p]_0, i \neq j, \bar{s} \in ([p]_0 \setminus \{i, j\})^{d-1} \},$$

each edge representing a legal move of disc  $d$  between pegs  $i$  and  $j$  with the  $d-1$  smaller discs in state  $\bar{s}$  and the  $n-d$  larger ones in state  $\underline{s}$ . Alternatively, the edge sets can be constructed recursively by

$$E(H_p^0) = \emptyset, \\ E(H_p^{1+n}) = \left\{ \{ir, is\}; i \in [p]_0, \{r, s\} \in E(H_p^n) \right\} \cup \left\{ \{ir, jr\} \in \binom{[p]_0^{1+n}}{2}; r \in [n] \subset [p]_0 \setminus \{i, j\} \right\}.$$

(The last set reflects the *divine rule* never to place a larger disc on a smaller one, which guarantees the natural order of the discs on a peg.) Note that  $H_p^1$  is isomorphic to  $K_p$ , the complete graph of order  $p$ .

It is immediate from this definition that the order of  $H_p^n$  is  $|H_p^n| = p^n$  and that the size fulfills the recurrence

$$\|H_p^0\| = 0, \quad \|H_p^{1+n}\| = p \|H_p^n\| + \binom{p}{2} (p-2)^n,$$

whence

$$\|H_p^n\| = \frac{1}{2} \binom{p}{2} (p^n - (p-2)^n).$$

The Hanoi graphs  $H_3^n$ , representing the original three-peg puzzle, are well understood. In particular, properties of the metric arising canonically from the connectedness of the graphs have been studied intensely. For instance, it is known that the diameter of  $H_3^n$  is  $2^n - 1$  [8, Theorem 1] and that there are at most two shortest paths between any two vertices [8, Theorem 4]. The number of tasks with non-unique shortest paths has been determined (cf. [10]). Moreover,  $H_3^n$  contains a perfect 1-code [6, Theorem 1]. Investigations about average distances in  $H_3^n$  (cf. [5], [8, Section 1.3.2]) led to the interpretation of the Sierpiński triangle as the limit of  $H_3^n$  for  $n \rightarrow \infty$  (cf. [13]).

For  $p > 3$ , questions about metric properties of  $H_p^n$  become notoriously difficult. Shortest paths can be highly non-unique (cf. [3]), diameters are not known in general and neither is the distance between two *perfect states*, i.e. states  $s = i^n$ , where all discs are on the same peg  $i \in [p]_0$ ; cf. [9,4,20]. In [14, Section 2.2], Hanoi graph  $H_p^n$  has been interpreted as a spanning subgraph of Hamming graph  $K_p^n$ .

A lot more is known about topological properties of  $H_p^n$ ,  $n \in \mathbb{N}$ , in general. They are hamiltonian [11, Theorem 1] and can be classified with respect to planarity [11, Theorem 2]. (Planarity can easily be proved by induction on  $n$  for  $p = 3$ ,  $H_4^1 \cong K_4$  is planar and so is  $H_4^2$  (see the drawing in Fig. 3), but  $H_4^3$  (and consequently  $H_4^n$ ,  $n > 3$ ) is not, a subdivision of  $K_{3,3}$  being induced, e.g., by the sets of vertices  $\{210, 102, 021\}$  and  $\{012, 201, 120\}$ ; finally,  $K_5 \cong H_5^1$  is contained in all  $H_p^n$ s for  $p > 4$ .) The connectivity is  $\kappa(H_p^n) = p-1$ : deleting all  $p-1$  neighbors of a perfect state will leave the latter isolated; to show that  $p-2$  vertices are *not* sufficient to disconnect  $H_p^n$ , we proceed by induction on  $n$ : for  $n = 1$ , i.e.  $H_p^1 \cong K_p$ , this is obvious and if we delete  $p-2$  vertices from  $H_p^{1+n}$ , at most  $(p-2)^2$  edges between subgraphs  $kH_p^n$ , induced by vertices of the form  $k\bar{s}$  with  $k \in [p]_0$  and  $\bar{s} \in [p]_0^n$ , are destroyed. But every pair of these subgraphs has  $(p-2)^n \geq p-2$  connecting edges, such that at most  $p-2$  connections between the subgraphs are completely lost, which leaves them still connected.

The chromatic numbers  $\chi(H_p^n)$  were found by Arett (cf. [2]; see Theorem 2). Moreover, it had been known that for  $n \in \mathbb{N}$  there is an essentially unique 3-edge-coloring for  $H_3^n$  given by the label of the *idle peg*, i.e. the one not involved in the move corresponding to an edge (cf. [23, p. 97]). However, the quest by Stockmeyer [26] for the chromatic index  $\chi'(H_p^n)$  for bases higher than 3 turned out to be quite intriguing by the fact that an answer seemed to be more demanding for small  $n$  and large  $p$  than for higher values of  $n$ , the case which usually is the more difficult one in connection with metric questions. The present paper will decide the edge- and total chromatic classification of  $H_p^n$  together with the analogous problem for the Sierpiński graphs  $S_p^n$ . For the latter, we will thereby refute a conjecture by Jakovac and Klavžar [15, Conjecture 4.5].

### 0.2. Sierpiński graphs

Sierpiński graphs were introduced in [17], motivated by topological studies on universal spaces based on relations on the sets  $V(S_p^n) = [p]_0^n$ . The graphs were named for Sierpiński because of direct connections to the Sierpiński triangle mentioned before. The edge sets are defined by

$$E(S_p^n) = \{ \{ \underline{s}i\bar{j}^{d-1}, \underline{s}j\bar{i}^{d-1} \}; d \in [n], \underline{s} \in [p]_0^{n-d}, i, j \in [p]_0, i \neq j \}$$

or can be given recursively by

$$E(S_p^0) = \emptyset, \\ E(S_p^{1+n}) = \left\{ \{ir, is\}; i \in [p]_0, \{r, s\} \in E(S_p^n) \right\} \cup \left\{ \{ij^n, ji^n\}; i, j \in [p]_0, i \neq j \right\}.$$

Again,  $S_p^1 \cong K_p$  for all  $p$ ,  $|S_p^n| = p^n$ , and for the size we have

$$\|S_p^0\| = 0, \quad \|S_p^{1+n}\| = p \|S_p^n\| + \binom{p}{2},$$

whence

$$\|S_p^n\| = \frac{p}{2}(p^n - 1).$$

Note that we may include the case  $p = 2$  without losing connectedness,  $S_2^n$  being isomorphic to the path graph  $P_{2^n}$  and thus also to the state graph of another famous puzzle, the *Chinese rings* (also known by the French name *Baguenaudier*) with  $n$  rings (cf. [1]). Moreover, Sierpiński graphs can be viewed as iterated complete graphs in the sense that  $S_p^{1+n}$  consists of  $p$  copies of  $S_p^n$  connected by the edges of a  $K_p$  with these copies as vertices. But there is also an interpretation in terms of pegs and discs, namely the STH, as shown in [17, Theorem 1]: as before, the vertex  $s = s_n \dots s_1$  of  $S_p^n$  represents the regular state of a puzzle with disc  $d$  on peg  $s_d$ , but now a move (edge) is the exchange of some disc  $d \in [n]$  on peg  $i$  with the *sub-tower* consisting of all discs in  $[d - 1]$  on peg  $j \neq i$ . For easy comparison with Hanoi graphs and since everything is known about path graphs, we will restrict  $p$  to values greater than 2 for Sierpiński graphs as well.

In [17, Theorem 2], it is proved that  $S_3^n \cong H_3^n$  (cf. also [13, Lemma 2] and [22, Theorem 2]), but looking at the sizes (or degrees; cf. Proposition 1), a short calculation shows that  $S_p^n$  and  $H_p^n$  are not isomorphic for  $p > 3$  and  $n > 1$ . Among the other properties of  $S_p^n$  proved in [17] are hamiltonicity [17, Proposition 3] (in [16, Theorem 4.3] it is shown that for  $S_3^n$ , and consequently  $H_3^n$ , the hamiltonian cycle is unique) and the fact that there are at most two shortest paths between any two vertices [17, Theorem 6]. (For  $p = 3$ , the decision whether to move the largest disc once or twice can be made by a finite automaton; cf. [22, Theorem 1].) Their distance, which is at most  $2^n - 1$  [17, Theorem 5], can be computed in  $O(n)$  time [17, Corollary 7], based on an extremely simple formula [17, Lemma 4] for the case of the distance between an arbitrary vertex  $s$  and an *extreme vertex*  $i^n$ , which is (using Iverson's convention ( $S$ ) = 1, if statement  $S$  is true, ( $S$ ) = 0, if  $S$  is false)  $\sum_{d=1}^n (s_d \neq i) \cdot 2^{d-1}$  (cf. also [13] for the case  $p = 3$  and [8, Theorem 3] for the corresponding result for  $H_3^n$ ). This is also the reason why the methods to determine average eccentricities and distances carry over from  $p = 3$  to larger  $p$  for Sierpiński, but not for Hanoi graphs (cf. [12]). Further details about metric properties of Sierpiński graphs can be found in [21], where one also finds the chromatic number  $\chi(S_p^n)$  (see Theorem 2). The chromatic index  $\chi'(S_p^n)$  was first known (with a unique minimal coloring) for  $p = 3$  (cf. [16, Corollary 3.3]) and was later determined for larger  $p$  as well (cf. [15, Theorem 4.1]). In the latter paper also some partial findings on total colorings of Sierpiński graphs (cf. [15, Propositions 4.2 to 4.4]) can be found. The result about (essentially unique) perfect 1-codes could be extended from  $H_3^n \cong S_3^n$  to  $S_p^n$  ([18, Theorem 3.6]; cf. also [7, Corollary 2.3]), whereas there are no perfect 1-codes for  $H_p^n$ , if  $p > 3$  and  $n > 2$  (cf. [25, Theorem 1]; for  $n = 2$ , the obviously only perfect 1-code consists of the perfect states).

The range of parameters  $p$  and  $n$  for which planarity of  $S_p^n$  holds is the same as for  $H_p^n$ . Again this is trivial for  $p > 4$ , follows by isomorphy for  $p = 3$  and  $S_4^1$ , and can be seen from the drawing of  $S_4^2$  in Fig. 4; a subdivision of  $K_{3,3}$  in  $S_4^3$  is obtained from the sets of vertices  $\{011, 022, 033\}$  and  $\{333, 000, 012\}$ , for instance. The non-planarity of  $S_4^n$  for  $n \geq 3$  also follows from [19, Proposition 3.2], where (positive) lower and upper bounds for the crossing-numbers are given. However, there is, to the best of our knowledge, no result on the genus of any non-planar Hanoi or Sierpiński graph other than for  $n = 1$  (Ringel and Youngs, 1968; cf. [28, Theorem 14D]). Connectivity is again  $\kappa(S_p^n) = p - 1$ ; the only difference from the argument for Hanoi graphs is that now at most  $p - 2$  edges between subgraphs  $kS_p^n$  are destroyed, but each pair of them is connected by exactly one edge.

In order to avoid the triviality of a static (S)TH, we exclude  $n = 0$  from the considerations which follow and begin with the well-known case  $n = 1$ .

## 1. Complete graphs

For the complete graphs  $K_p$  the following facts are well-known (cf. [28, Theorem 20B], [29, Theorem 3.1]); we will give the proofs anyway in order to introduce specific colorings needed in later sections.

**Theorem 1.**  $\delta(K_p) = p - 1 = \Delta(K_p)$ ,  
 $\chi(K_p) = p$ ,  $\chi'(K_p) = p - (p \text{ even})$ ,  $\chi''(K_p) = p + (p \text{ even})$ .

**Proof.** Degrees and vertex-chromatic number  $\chi$  are clear. The *canonical vertex-coloring*  $c_p$  will be given by  $\forall i \in [p]_0 : c_p(i) := i$ .

| $i \setminus j$ | 0 | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|---|
| 0               |   |   | 3 | 1 | 4 |
| 1               | 3 |   | 4 | 2 | 0 |
| 2               | 1 | 4 |   | 0 | 3 |
| 3               | 4 | 2 | 0 |   | 1 |
| 4               | 2 | 0 | 3 | 1 |   |

Fig. 1. Canonical 5-edge-coloring for  $K_5$ .

| $i \setminus j$ | 0 | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|---|
| 0               | 0 | 3 | 4 | 2 | 1 |
| 1               | 3 | 1 | 0 | 4 | 2 |
| 2               | 4 | 0 | 2 | 1 | 3 |
| 3               | 2 | 4 | 1 | 3 | 0 |
| 4               | 1 | 2 | 3 | 0 | 4 |

Fig. 2. Special total 5-coloring for  $K_5$ .

Non-trivial lower bounds for the edge-chromatic number  $\chi'$  in the odd case and for the total chromatic number  $\chi''$  in the even case are provided by

$$\binom{p}{2} = \|K_p\| \leq \frac{p-1}{2} \chi'(K_p) \quad \text{and} \quad \binom{p+1}{2} = |K_p| + \|K_p\| \leq \frac{p}{2} \chi''(K_p),$$

respectively. (By the pigeonhole principle, one color cannot paint  $\frac{p+1}{2}$  edges, involving  $p+1$  end-vertices, if  $p$  is odd; similarly, for even  $p$  each color can cover at most  $\frac{p}{2}$  edges and no vertex or  $\frac{p}{2} - 1$  edges and one vertex.)

The upper bounds for  $\chi'$  and  $\chi''$  will be based on an explicit edge-coloring which we will call the *canonical edge-coloring*  $c'_p$ .

To construct  $c'_p$ , we start with an odd  $p$ . For  $i, j \in [p]_0$ ,  $i \neq j$ , edge  $\{i, j\}$  has color  $c'_p(i, j) := \frac{i(p+1)-j(p-1)}{2} \bmod p$ . (Since we are dealing with undirected edges only, we will write  $c(i, j)$  when talking about the color assigned to edge  $\{i, j\}$  in an edge-coloring  $c$  throughout.) This coloring has the property that in every vertex  $k \in [p]_0$  all  $p$  colors from  $[p]_0$  are present except color  $k$  (cf. Fig. 1), which follows from  $i + j - 2c'_p(i, j) \in \{-p, 0, p\}$ . This allows us to join vertex  $k$  with a further vertex  $p$  by an edge colored by  $k$ , thus obtaining the canonical coloring for  $K_{p+1}$ . (Note that edges of the same color make up a perfect matching thus leading by induction to perfect matchings for all Hanoi and Sierpiński graphs of even base.)

A total  $p$ -coloring for  $K_p$  can now be constructed for odd  $p$  by simply combining  $c_p$  and  $c'_p$ ; leaving out line  $p - 1$  will then give a total  $p$ -coloring for  $K_{p-1}$ .  $\square$

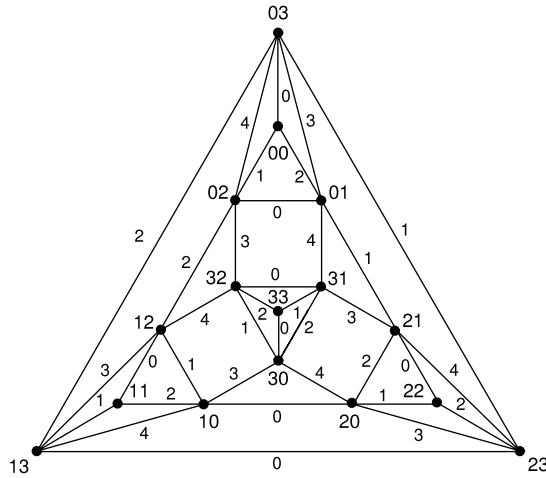
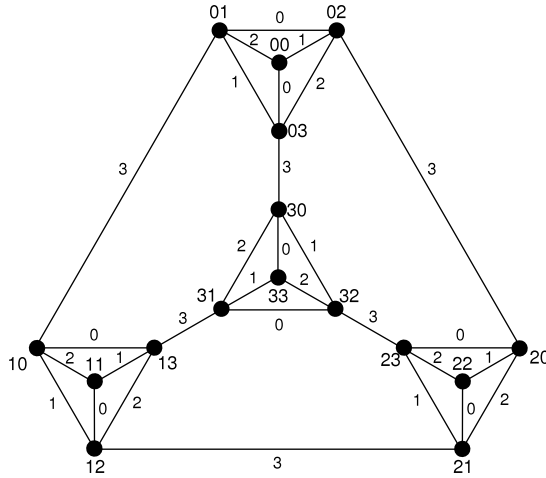
The *canonical total coloring* just constructed has the disadvantage, for even  $p$ , that the color missing at vertex  $k$  is somewhat unorderedly related to  $k$ . For later purpose we will therefore introduce a *special edge-coloring*  $c''_p$ , for which we use the same approach as for  $c'_p$ , but with the roles of odd and even switched. For  $k \in [p]_0$ , let  $\tau_k$  be the transposition of  $k$  and  $p - 1$  on  $[p]_0$ . Then, for even  $p$ ,  $c''_p(i, j) := (\tau_i(j) + 2 + \tau_j(i)) \bmod (p + 1)$  for  $i, j \in [p]_0$ ,  $i \neq j$ , defines a  $(p + 1)$ -edge-coloring with colors  $k$  and  $(k + 1) \bmod p$  missing in line  $k \in [p]_0$ . Therefore, we may color the vertices by the canonical vertex-coloring to obtain the *special total*  $(p + 1)$ -coloring of  $K_p$ . Since color  $(k + 1) \bmod p$  is still missing in line  $k$ , we can join vertex  $k$  with a new vertex named and colored  $p$  with an edge of this color, thus obtaining the special edge-/total  $(p + 1)$ -coloring for  $K_{p+1}$  (cf. Fig. 2).

In view of Theorem 1 we may assume that  $n \geq 2$  from now on.

## 2. Vertex-colorings

The result on the chromatic numbers of Hanoi and Sierpiński graphs mentioned in the Introduction reads as follows.

**Theorem 2.**  $\chi(H_p^n) = p = \chi(S_p^n)$ .

Fig. 3. Hanoi graph  $H_4^2$  with 5-edge-coloring.Fig. 4. Sierpiński graph  $S_4^2$  with 4-edge-coloring.

**Proof.** Since  $H_p^1$  is (embedded as) a subgraph of  $H_p^n$ , it is clear that  $\chi(H_p^n) \geq p$  and similarly for the Sierpiński case. From the definitions of  $H_p^n$  and  $S_p^n$ , we have:

A  $p$ -coloring for  $H_p^n$  is given by

$$[p]_0^n \ni s \mapsto \left( \sum_{d=1}^n s_d \right) \bmod p \in [p]_0,$$

because every move in the TH consists of the change of the value of precisely one  $s_d$ .

A  $p$ -coloring for  $S_p^n$  is given by

$$[p]_0^n \ni s \mapsto s_1 \in [p]_0,$$

because every move in the STH involves the change of the position of disc 1.  $\square$

### 3. Edge-colorings

We begin our analysis of edge-colorings with formulas for extremal degrees.

**Proposition 1.** The minimal degrees are

$$\delta(H_p^n) = p - 1 = \delta(S_p^n),$$

and the maximal degrees are

$$\Delta(H_p^n) = \binom{p}{2} - \binom{p-n}{2}, \quad \Delta(S_p^n) = p$$

$$(\binom{p-n}{2} := 0, \text{ for } n > p).$$

**Proof.** The divine rule allows up to  $k \in [p-1]$  topmost discs to move in a given state of the TH. The smallest of these can move to  $p-1$  pegs, the second smallest to  $p-2$  & c. This adds up to

$$\sum_{l=1}^k (p-l) = \sum_{l=1}^{p-1} (p-l) - \sum_{l=k+1}^{p-1} (p-l) = \sum_{l=1}^{p-1} l - \sum_{l=1}^{p-k-1} l = \binom{p}{2} - \binom{p-k}{2}.$$

The minimal degree is obtained for  $k=1$  and there are states with  $k=p-1$ , if  $n \geq p-1$ ; otherwise  $k$  can only be as large as  $n$ .

In a given state of the STH, disc 1 can always move to  $p-1$  destinations, and one more move is allowed involving the maximal subtower it belongs to in a non-extreme vertex.  $\square$

An immediate consequence of [Proposition 1](#) is the following result which will be useful later on.

**Corollary 1.** For  $p > 3$ :  $\Delta(H_p^n) = \Delta(H_{p-1}^{n-1}) + p - 1$ .

### 3.1. Main theorem

Our comprehensive result on edge-colorings is the following.

**Theorem 3.**  $\chi'(H_p^n) = \Delta(H_p^n)$ ,  $\chi'(S_p^n) = \Delta(S_p^n)$ .

**Remark.** With the edge-coloring classes based on Vizing's Theorem (cf. [27, p. 277f]) this means that all Hanoi and Sierpiński graphs are class 1 except the instances of  $H_p^1$  and  $S_p^1$  with odd  $p$ .  $\square$

As it is clear that  $\chi'(G) \geq \Delta(G)$  for any graph, we will prove our theorem by providing appropriate  $\Delta(G)$ -colorings in five steps, which will be performed in the next subsection. (In what follows, a “ $k$ -coloring” will always mean a proper  $k$ -edge-coloring (cf. [27, Definition 7.1.3]).)

**Lemma 1.** For  $n \geq p-1$ ,  $\chi'(H_p^n) \leq \binom{p}{2} (= \Delta(H_p^n))$ .

**Lemma 2.**  $\chi'(H_p^2) \leq 2p-3 (= \Delta(H_p^2))$ .

**Lemma 3.** For  $p > 3$ ,  $\chi'(H_p^n) \leq \chi'(H_{p-1}^{n-1}) + p - 1$ .

**Lemma 4.** For  $p$  even,  $S_p^n$  can be edge-colored with colors from  $[p]_0$  (with edges colored with color  $p-1$  not incident with extreme vertices).

**Lemma 5.** For  $p$  odd,  $S_p^n$  can be edge-colored with colors from  $[p]_0$  with edges colored with color  $k \in [p]_0$  not incident with extreme vertex  $k^n$ .

**Proof of Theorem 3.** The Hanoi case follows by double induction on  $p \geq 3$  and  $n \geq 2$  from [Lemma 1](#) (applied for  $p=3$ ), [Lemma 2](#) (to cover  $n=2$ ), and [Lemma 3](#) together with [Corollary 1](#). The Sierpiński case is obvious from [Lemmas 4](#) and [5](#) in view of [Proposition 1](#).  $\square$

### 3.2. Proofs of the lemmas

**Proof of Lemma 1.** Let us color the edges of  $H_p^n$  according to the (unordered) pair of pegs involved in the corresponding move of a disc. There are  $\binom{p}{2}$  such pairs. By the divine rule, all edges coming together in a fixed state correspond to moves with different pairs of pegs. Thus we have obtained a  $\binom{p}{2}$ -coloring, which is a  $\Delta(H_p^n)$ -coloring by [Proposition 1](#).  $\square$

**Proof of Lemma 2.** The case  $p=3$  is already included in [Lemma 1](#).

For even  $p$ , the following algorithm will produce a  $(2p-3)$ -coloring.

**Step 0.** Write down the adjacency matrix  $A$  for  $H_p^2$  with the vertices arranged in natural order if we interpret  $s \in V(H_p^n)$  as a number in the number system of base  $p$ . The coloring will now be constructed by filling the non-zero entries of  $A$  with colors ranging from 0 to  $2(p-2)$ .

**Step 1.** For  $k \in [p]_0$ , the  $(p-1) \times (p-1)$  sub-matrix of  $A$  induced by the set of vertices  $\{kl; l \in [p-1]_0\}$  represents a complete subgraph  $K_{p-1}$  of  $H_p^2$  which can be colored according to the canonical coloring from the proof of [Theorem 1](#) using colors from  $[p-1]_0$ .

**Step 2.** For  $k \in [p]_0, l \in [p-1]_0$ , the edges  $\{kl, k(p-1)\}$  are colored with color  $l$ , if  $k = p-1$  and otherwise with color  $p-2+l-k$  for  $k \in [l]_0$ , color  $l$  for  $k = l$ , and color  $2p-3+l-k$  for  $l \in [k]_0$ . (This amounts to say that for  $k \neq p-1$  the moves from  $kl$  to  $k(p-1)$  form a latin square of order  $p-1$  with the symbols from  $[2(p-2)] \setminus [p-3]$ , if we replace the entries in the diagonal ( $\{kk, k(p-1)\}$ ) with the symbol  $p-2$ .)

**Step 3.0.** The sub-matrix of  $A$  induced by the set of vertices  $\{k(p-1); k \in [p-1]_0\}$  can be filled as a  $K_{p-1}$  using colors from  $[p-1]_0$  in the canonical way.

**Step 3.1.** The remaining  $p-1$  sub-matrices  $A_l$  of  $A$  induced by the sets of vertices  $\{kl; k \in [p]_0 \setminus \{l\}\}$  for  $l \in [p-1]_0$  have to be filled as  $K_{p-1}$ s using colors from  $\{l, p-1, \dots, 2(p-2)\}$ , avoiding the use of color  $l$  for vertices  $(p-1)l$  and of colors  $p-2+l-k$  for  $k \in [l]_0$  and  $2p-3+l-k$  for  $l \in [k]_0$ , if  $k \neq p-1$ . (This can be achieved since precisely one of the colors is missing in each row of  $A_l$ , just like precisely color  $q$  is missing in the  $q$ th row of the canonically colored adjacency matrix for  $K_{p-1}$ .)

To prove correctness of this algorithm, we note that after Step 2 all moves of disc 1 and only these are covered. The subgraphs in Step 3 are mutually independent because the position of disc 1 is fixed (at  $p-1$  in 3.0, at  $l$  in 3.1) in each of them. The choice of colors in Step 3 is such that it is compatible with Step 2.

The execution of this algorithm can be followed easily for the example of  $H_4^2$ ; cf. [Fig. 3](#) and the corresponding adjacency matrices in the [Appendix](#).

The remaining case of odd  $p > 3$  is a consequence of [Theorem 1](#) and [Lemma 3](#) to be proved now.  $\square$

**Proof of Lemma 3.** We start by filling the adjacency matrix of  $H_p^n$  along the main diagonal with  $p^{n-1}$  identical copies of the canonically colored  $K_p$ . This already covers all moves of disc 1. The rest of the edges can be decomposed into  $p$  mutually independent subgraphs  $H(l)$ , each with a fixed position  $l \in [p]_0$  of disc 1. Only  $p-1$  colors are used for moves of disc 1 to or from peg  $l$  ( $[p]_0 \setminus \{l\}$ , if  $p$  is odd,  $[p-1]_0$ , if  $p$  is even). To prove [Lemma 3](#), we therefore have  $\chi'(H_{p-1}^{n-1})$  colors free to color  $H(l)$ .

The graph  $H(l)$  in turn decomposes into mutually independent subgraphs depending on what lies below disc 1 on peg  $l$  and is therefore blocked inside  $H(l)$ . The worst case with respect to edge-coloring is when disc 1 is alone on peg  $l$ . With disc 1 and peg  $l$  missing, the corresponding component of  $H(l)$  is isomorphic to  $H_{p-1}^{n-1}$ .  $\square$

**Remark.** It is not possible to employ the strategy in the proof of [Lemma 3](#) for the algorithm in the proof of [Lemma 2](#), namely to use  $p$  identical canonically colored copies of  $K_p$  along the diagonal of the adjacency matrix for  $H_p^2$  when  $p$  is even. The reason is that this would use up all  $p-1$  colors from  $[p-1]_0$ , and another  $\chi'(H_{p-1}^1) = p-1$  would be necessary to color the  $H(l)$ s, altogether  $2(p-1)$  instead of  $\Delta(H_p^2) = 2p-3$  colors. This is also clear from [Fig. 3](#), where one would not be able to join the vertices 03, 13, and 23 by properly colored edges.

On the other hand, it should be noted that a proper coloring of  $H_4^2$  can be obtained from the coloring of  $H_3^2$  realized in the proof of [Lemma 2](#) (see the subgraph induced by vertices without a 3 in their labels in [Fig. 3](#)) using 2 extra colors. This is not possible for the step from  $H_5^2$  to  $H_6^2$ , however, such that the latter turns out to be the most crucial case which has to be treated by the algorithm in the proof of [Lemma 2](#).  $\square$

For the Sierpiński case as well it is necessary to proceed according to the parity of  $p$  due to the different behavior of  $K_p$  with respect to edge-coloring. Here it turns out that the even case ([Lemma 4](#)) is easy (cf. [Fig. 4](#)) while the odd case ([Lemma 5](#)) is slightly more demanding.

**Proof of Lemma 4.** Edges corresponding to the switching of a disc and a non-empty sub-tower in the STH are colored with color  $p-1$ , while those belonging to individual moves of disc 1 from  $i$  to  $j$ , say, receive colors from  $[p-1]_0$  according to a scheme produced from the colored adjacency matrix of  $K_p$  by interpreting the rows as index  $i$  and the columns as index  $j$  (cf. the adjacency matrices in the [Appendix](#)).  $\square$

Because of the extra color needed for  $K_p$  if  $p$  is odd, induction is unavoidable for the proof of [Lemma 5](#).

**Proof of Lemma 5.** For  $n=1$  the statement follows from  $S_p^1 \cong K_p$  and the canonical coloring of the latter in the proof of [Theorem 1](#). For the induction step we first color the edges  $\{ij^n, ji^n\}$  according to the colored adjacency matrix of  $K_p$  (cf. [Fig. 1](#) and the [Appendix](#)). By the special form of the canonical coloring of  $K_p$  and induction assumption, we can place  $p$  copies of  $S_p^n$  along the diagonal of the adjacency matrix for  $S_p^{1+n}$  with color  $k$  fixed for the copy representing  $r_{1+n} = k$  and the other  $p-1$  different colors  $c$  permuted in each copy in such a way that these colors do not conflict with those already assigned before. (This permutation is given by  $c \mapsto \frac{c(p+1)-k(p-1)}{2} \bmod p$ ; cf. the [Appendix](#).)  $\square$



#### 4. Total colorings

The total chromatic numbers of Hanoi and Sierpiński graphs can be obtained in a similar fashion as the chromatic indices. However, as can be seen from [Theorem 1](#), the roles of odd and even  $p$  change. Again it is clear that  $\chi'' \geq \Delta + 1$ , but the analogue to Vizing's theorem for the chromatic index, namely  $\chi'' \leq \Delta + 2$ , known as the *Vizing–Behzad conjecture* (cf. [24, Section 16.2]), is not yet proved for all graphs. In our case, however, we can construct  $(\Delta + 1)$ -total colorings.

**Theorem 4.**  $\chi''(H_p^n) = \Delta(H_p^n) + 1$ ,  $\chi''(S_p^n) = \Delta(S_p^n) + 1$ .

The proof will again be divided into several lemmas.

##### 4.1. Total Hanoi colorings

The hitch with total Hanoi colorings is that no provable analogue of [Lemma 1](#), namely a natural total  $(\Delta + 1)$ -coloring if  $n \geq p - 1$ , seems to be at hand. So we have to be modest and show

**Lemma 1'.**  $\chi''(H_3^n) \leq 4 (= \Delta(H_3^n) + 1)$ .

**Proof.** Each of the  $3^{n-1}$  subgraphs of  $H_3^n$  belonging to fixed distributions of discs 2 to  $n$  can be totally colored using the three colors 0, 1, and 2 according to [Theorem 1](#). (Actually this can be achieved respecting the vertex-coloring from [Theorem 2](#).) The remaining edges, belonging to moves of discs 2 to  $n$ , can then all be colored with color 3.  $\square$

We also get an analogue of [Lemma 2](#), namely

**Lemma 2'.**  $\chi''(H_p^2) \leq 2(p - 1) (= \Delta(H_p^2) + 1)$ .

**Proof.** The case  $p = 3$  is already covered by [Lemma 1'](#).

For even  $p$ , we proceed by the following steps (cf. [Appendix](#) for the case  $p = 4$ ).

1. Color edges of  $p$  copies of  $K_p$  on the diagonal of the adjacency matrix of  $H_p^2$  (moves of disc 1) identically according to the special coloring  $c_p'$ . Two of the colors from  $[p + 1]_0$  are missing at vertices ending in  $k \in [p]_0$ , namely  $k$  and  $(k + 1) \bmod p$ .
2. Color vertex  $ij$ ,  $i, j \in [p]_0$ , according to the following scheme: let  $h = (j - i) \bmod p$ , then

$$c(ij) = \begin{cases} j, & h = 0 \text{ or } h = p - 1; \\ (j + 1) \bmod p, & h = 1; \\ p - 1 + h, & \text{otherwise.} \end{cases}$$

It is not difficult to convince oneself that this is a proper vertex-coloring compatible with step 1.

3. For  $k \in [p]_0$ , color the subgraph induced by vertices  $lk$ ,  $l \in [p]_0 \setminus \{k\}$  (moves of disc 2 with disc 1 fixed on bottom of peg  $k$ ) totally as in the special total coloring of  $K_{p-1}$ , but respecting the vertex-coloring from step 2 and using colors  $k$ ,  $(k + 1) \bmod p$  and  $\chi''(K_{p-1}) - 2 = p - 3$  new colors from  $[2(p - 1)]_0 \setminus [p + 1]_0$ .

The case of odd  $p > 3$  will be included in the following lemma.  $\square$

**Lemma 3'.** (a) For odd  $p > 3$ ,  $\chi''(H_p^n) \leq \chi'(H_{p-1}^{n-1}) + p$ .

(b) For even  $p$ ,  $\chi''(H_p^n) \leq \chi''(H_{p-1}^{n-1}) + p - 1$ .

**Proof.** (a) Color vertices of  $H_p^n$  according to the coloring  $c$  of [Theorem 2](#). The  $p^{n-1}$  subgraphs of  $H_p^n$  corresponding to fixed distributions of discs 2 to  $n$  can then be canonically total-colored as  $K_p$  but respecting the vertex-coloring already prescribed. (The color of edge  $\{u, v\}$  is given by  $c_p'(c(u), c(v))$ .) This uses all colors from  $[p]_0$ . The rest of the edges can be colored using  $\chi'(H_{p-1}^{n-1})$  extra colors as in [Lemma 3](#). For the example of  $H_5^2$ , see the [Appendix](#).

(b) We may assume  $n > 2$ , the case  $n = 2$  being covered by [Lemma 2'](#). We start by coloring vertices according to [Theorem 2](#). The subgraph of  $H_p^n$  remaining after deletion of all edges corresponding to disc-1-moves decomposes into  $p$  mutually disconnected subgraphs  $H(l)$  induced by the position  $l \in [p]_0$  of disc 1. Let us consider  $H(p - 1)$ . It is in turn made up from  $2^{n-1}$  mutually unconnected components, each given by a subset  $M$  of discs lying on peg  $p - 1$  (and therefore being blocked by disc 1). The component belonging to  $M$  is isomorphic to  $H_{p-1}^{n-m}$ ,  $m := |M| \in [n]$ . (Obviously, there are  $\binom{n-1}{m-1}$  such components.) As we saw in the proof of [Lemma 1'](#) and in part (a) of this proof, these components can be optimally total-colored respecting the vertex-colors already prescribed, because  $p - 1$  is odd. The worst case is  $m = 1$ , such that all colors from  $[\gamma]_0$  with  $\gamma := \chi''(H_{p-1}^{n-1})$  are used. In a similar way, all the other  $H(l)$ s can be total-colored employing the same  $\gamma$  colors and respecting the vertex-coloring, which is possible because  $\gamma \geq p = \chi(H_p^n)$  since  $n > 2$ . (This is the reason why we had to treat the case  $n = 2$  separately. Actually, we are using an induction argument here already.) Now all that



|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 4 | 5 | 6 | 2 |
| 1 | 3 | 2 | 5 | 6 | 0 | 4 |
| 2 | 4 | 5 | 3 | 0 | 1 | 6 |
| 3 | 5 | 6 | 0 | 4 | 2 | 1 |
| 4 | 6 | 0 | 1 | 2 | 5 | 3 |
| 5 | 2 | 4 | 6 | 1 | 3 | 0 |

Fig. 5. 7-total-coloring for  $K_6$ .

|   | 0       | 1       | 2       | 3       | 4       | 5       |
|---|---------|---------|---------|---------|---------|---------|
| 0 | $\Pi_0$ | 3       | 4       | 5       | 6       | 2       |
| 1 | 3       | $\Pi_1$ | 5       | 6       | 0       | 4       |
| 2 | 4       | 5       | $\Pi_2$ | 0       | 1       | 6       |
| 3 | 5       | 6       | 0       | $\Pi_3$ | 2       | 1       |
| 4 | 6       | 0       | 1       | 2       | $\Pi_4$ | 3       |
| 5 | 2       | 4       | 6       | 1       | 3       | $\Pi_5$ |

Fig. 6. 7-total-coloring for  $S_6^2$ .

remains is to color the edges belonging to disc-1-moves with extra colors. Since  $p$  is even, this can be done with  $p - 1$  colors by Theorem 1.  $\square$

The Hanoi case of Theorem 4 now follows immediately from these lemmas by induction.

#### 4.2. Total Sierpiński colorings

The Sierpiński case of Theorem 4 is a direct consequence of the following two lemmas.

**Lemma 4'.** For  $p$  odd,  $S_p^n$  can be totally colored with colors from  $[p + 1]_0$  (with edges colored with color  $p$  not incident with extreme vertices).

**Lemma 5'.** For  $p$  even,  $S_p^n$  can be totally colored with colors from  $[p + 1]_0$  with edges colored with color  $k \in [p]_0$  not incident with extreme vertex  $k^n$ .

**Proof of Lemma 4'.** As for Lemma 4, we color all edges corresponding to non-trivial switches with color  $p$  and all subgraphs belonging to fixed distributions of discs 2 to  $n$  according to the special total coloring of  $K_p$ , using colors from  $[p]_0$  (cf. the adjacency matrix for  $S_5^2$  in the Appendix).  $\square$

**Proof of Lemma 5'.** As for Lemma 5, the proof is by induction on  $n \in \mathbb{N}$ . For  $n = 1$ , we can use the special total coloring for  $K_p \sim S_p^1$ , where we replace the canonical vertex-coloring by  $i \mapsto (i + 1) \bmod p$ .

For the induction step, the edges  $\{ij^n, ji^n\}$  are colored according to the specially colored adjacency matrix of  $K_p$ . The permutations of colors in the  $p$  copies of  $S_p^n$  along the diagonal of the adjacency matrix for  $S_p^{1+n}$  are then obtained by  $c \mapsto c_p''(c, k)$ ,  $c \in [p + 1]_0$ ,  $k \in [p]_0$ , where  $c_p''(k, k) := k$  and  $c_p''(p, k) := c_{p+1}'(p, k)$  (cf. the adjacency matrix for  $S_4^2$  in the Appendix).  $\square$

We finally give one more example, namely a proper total coloring with 7 colors for  $S_6^2$ , which is the smallest counterexample for [15, Conjecture 4.5]. We start with the special edge-coloring of  $K_6$  with vertices colored as in the preceding proof; see Fig. 5. Note that color  $k$  is missing in row  $k$ , i.e. at vertex  $k$ . One now obtains a 7-total-coloring of  $S_6^2$  from the scheme of Fig. 6. Again this is meant as a colored adjacency matrix, but in an abbreviated form, where  $k$  stands for the

subgraph with the larger disc on peg  $k$ . So the entry  $\Pi_k$  in row  $k$  and column  $k$  stands for the  $K_6$  of moves of disc 1 with disc 2 fixed on peg  $k$ . Here  $\Pi_k$  is obtained from the total coloring of  $K_6$  by the permutations:

$$\begin{aligned}\Pi_0 &= 0345621, & \Pi_1 &= 3156042, & \Pi_2 &= 4520163, \\ \Pi_3 &= 5603214, & \Pi_4 &= 6012435, & \Pi_5 &= 2461350.\end{aligned}$$

All the other entries in positions  $(k, l)$  stand for the switches of discs 1 and 2 between pegs  $k$  and  $l$ . Note that these entries again form the special edge-coloring of  $K_6$ !

## Acknowledgments

We thank Robert E. Jamison (Clemson) and Karin Wales (München) for valuable discussions, Sandi Klavžar (Ljubljana) for useful comments, Andreas Groh (München) for some computer experiments, and Wolfgang Sobetzki (München) for technical support.

## Appendix

Adjacency matrix with edge-coloring for  $H_4^2$  in the proof of Lemma 2 after Steps 0, 1, 2, and 3, respectively.

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | ×  | ×  | ×  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | ×  |    | ×  | ×  |    |    |    |    |    | ×  |    |    |    | ×  |    |    |
| 02 | ×  | ×  |    | ×  |    |    | ×  |    |    |    |    |    |    |    | ×  |    |
| 03 | ×  | ×  | ×  |    |    |    |    | ×  |    |    |    | ×  |    |    |    |    |
| 10 |    |    |    |    |    | ×  | ×  | ×  | ×  |    |    |    | ×  |    |    |    |
| 11 |    |    |    |    | ×  |    | ×  | ×  |    |    |    |    |    |    |    |    |
| 12 |    |    | ×  |    | ×  | ×  |    | ×  |    |    |    |    |    |    | ×  |    |
| 13 |    |    |    | ×  | ×  | ×  | ×  |    |    |    |    | ×  |    |    |    |    |
| 20 |    |    |    |    | ×  |    |    |    |    | ×  | ×  | ×  | ×  |    |    |    |
| 21 |    | ×  |    |    |    |    |    |    | ×  |    | ×  | ×  |    | ×  |    |    |
| 22 |    |    |    |    |    |    |    |    | ×  | ×  |    | ×  |    |    |    |    |
| 23 |    |    |    | ×  |    |    |    | ×  | ×  | ×  | ×  |    |    |    |    |    |
| 30 |    |    |    |    | ×  |    |    |    | ×  |    |    |    |    | ×  | ×  | ×  |
| 31 |    | ×  |    |    |    |    |    |    |    | ×  |    |    | ×  |    | ×  | ×  |
| 32 |    |    | ×  |    |    |    | ×  |    |    |    |    |    | ×  | ×  |    | ×  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | ×  | ×  | ×  |    |

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | 2  | 1  | ×  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 2  |    | 0  | ×  |    |    |    |    |    | ×  |    |    |    | ×  |    |    |
| 02 | 1  | 0  |    | ×  |    |    | ×  |    |    |    |    |    |    |    | ×  |    |
| 03 | ×  | ×  | ×  |    |    |    |    | ×  |    |    |    | ×  |    |    |    |    |
| 10 |    |    |    |    |    | 2  | 1  | ×  | ×  |    |    |    | ×  |    |    |    |
| 11 |    |    |    |    | 2  |    | 0  | ×  |    |    |    |    |    |    |    |    |
| 12 |    |    | ×  |    | 1  | 0  |    | ×  |    |    |    |    |    |    | ×  |    |
| 13 |    |    |    | ×  | ×  | ×  | ×  |    |    |    |    | ×  |    |    |    |    |
| 20 |    |    |    |    | ×  |    |    |    |    | 2  | 1  | ×  | ×  |    |    |    |
| 21 |    | ×  |    |    |    |    |    |    | 2  |    | 0  | ×  |    | ×  |    |    |
| 22 |    |    |    |    |    |    |    |    | 1  | 0  |    | ×  |    |    |    |    |
| 23 |    |    |    | ×  |    |    |    | ×  | ×  | ×  | ×  |    |    |    |    |    |
| 30 |    |    |    |    | ×  |    |    |    | ×  |    |    |    |    | 2  | 1  | ×  |
| 31 |    | ×  |    |    |    |    |    |    |    | ×  |    |    | 2  |    | 0  | ×  |
| 32 |    |    | ×  |    |    |    | ×  |    |    |    |    |    | 1  | 0  |    | ×  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | ×  | ×  | ×  |    |

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | 2  | 1  | 0  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 2  |    | 0  | 3  |    |    |    |    |    | ×  |    |    |    | ×  |    |    |
| 02 | 1  | 0  |    | 4  |    |    | ×  |    |    |    |    |    |    |    | ×  |    |
| 03 | 0  | 3  | 4  |    |    |    |    | ×  |    |    |    | ×  |    |    |    |    |
| 10 |    |    |    |    | 2  | 1  | 4  | ×  |    |    |    |    | ×  |    |    |    |
| 11 |    |    |    |    | 2  | 0  | 1  |    |    |    |    |    |    |    |    |    |
| 12 |    |    | ×  |    | 1  | 0  | 3  |    |    |    |    |    |    |    | ×  |    |
| 13 |    |    |    | ×  | 4  | 1  | 3  |    |    |    |    | ×  |    |    |    |    |
| 20 |    |    |    |    | ×  |    |    |    | 2  | 1  | 3  | ×  |    |    |    |    |
| 21 |    | ×  |    |    |    |    |    |    | 2  | 0  | 4  |    |    | ×  |    |    |
| 22 |    |    |    |    |    |    |    |    | 1  | 0  | 2  |    |    |    |    |    |
| 23 |    |    |    | ×  |    |    |    | ×  | 3  | 4  | 2  |    |    |    |    |    |
| 30 |    |    |    |    | ×  |    |    |    | ×  |    |    |    |    | 2  | 1  | 0  |
| 31 |    | ×  |    |    |    |    |    |    |    | ×  |    |    | 2  |    | 0  | 1  |
| 32 |    |    | ×  |    |    |    | ×  |    |    |    |    |    | 1  | 0  |    | 2  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | 0  | 1  | 2  |    |

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | 2  | 1  | 0  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 2  |    | 0  | 3  |    |    |    |    |    | 1  |    |    |    | 4  |    |    |
| 02 | 1  | 0  |    | 4  |    |    | 2  |    |    |    |    |    |    |    | 3  |    |
| 03 | 0  | 3  | 4  |    |    |    |    | 2  |    |    |    | 1  |    |    |    |    |
| 10 |    |    |    |    | 2  | 1  | 4  | 0  |    |    |    |    | 3  |    |    |    |
| 11 |    |    |    |    | 2  | 0  | 1  |    |    |    |    |    |    |    |    |    |
| 12 |    |    | 2  |    | 1  | 0  | 3  |    |    |    |    |    |    |    | 4  |    |
| 13 |    |    |    | 2  | 4  | 1  | 3  |    |    |    |    | 0  |    |    |    |    |
| 20 |    |    |    |    | 0  |    |    |    | 2  | 1  | 3  | 4  |    |    |    |    |
| 21 |    | 1  |    |    |    |    |    |    | 2  | 0  | 4  |    |    | 3  |    |    |
| 22 |    |    |    |    |    |    |    |    | 1  | 0  | 2  |    |    |    |    |    |
| 23 |    |    |    | 1  |    |    |    | 0  | 3  | 4  | 2  |    |    |    |    |    |
| 30 |    |    |    |    | 3  |    |    |    | 4  |    |    |    |    | 2  | 1  | 0  |
| 31 |    | 4  |    |    |    |    |    |    |    | 3  |    |    | 2  |    | 0  | 1  |
| 32 |    |    | 3  |    |    |    | 4  |    |    |    |    |    | 1  | 0  |    | 2  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | 0  | 1  | 2  |    |

Adjacency matrix with edge-coloring for  $S_4^2$  in the proof of [Lemma 4](#).

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | ×  | ×  | ×  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | ×  |    | ×  | ×  | 3  |    |    |    |    |    |    |    |    |    |    |    |
| 02 | ×  | ×  |    | ×  |    |    |    |    | 3  |    |    |    |    |    |    |    |
| 03 | ×  | ×  | ×  |    |    |    |    |    |    |    |    |    | 3  |    |    |    |
| 10 |    | 3  |    |    |    | ×  | ×  | ×  |    |    |    |    |    |    |    |    |
| 11 |    |    |    |    | ×  |    | ×  | ×  |    |    |    |    |    |    |    |    |
| 12 |    |    |    |    | ×  | ×  |    | ×  |    | 3  |    |    |    |    |    |    |
| 13 |    |    |    |    | ×  | ×  | ×  |    |    |    |    |    |    | 3  |    |    |
| 20 |    |    | 3  |    |    |    |    |    |    | ×  | ×  | ×  |    |    |    |    |
| 21 |    |    |    |    |    |    | 3  |    | ×  |    | ×  | ×  |    |    |    |    |
| 22 |    |    |    |    |    |    |    |    | ×  | ×  |    | ×  |    |    |    |    |
| 23 |    |    |    |    |    |    |    |    | ×  | ×  | ×  |    |    |    | 3  |    |
| 30 |    |    |    | 3  |    |    |    |    |    |    |    |    |    | ×  | ×  | ×  |
| 31 |    |    |    |    |    |    |    | 3  |    |    |    |    | ×  |    | ×  | ×  |
| 32 |    |    |    |    |    |    |    |    |    |    |    | 3  | ×  | ×  |    | ×  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | ×  | ×  | ×  |    |

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | 2  | 1  | 0  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 2  |    | 0  | 1  | 3  |    |    |    |    |    |    |    |    |    |    |    |
| 02 | 1  | 0  |    | 2  |    |    |    |    | 3  |    |    |    |    |    |    |    |
| 03 | 0  | 1  | 2  |    |    |    |    |    |    |    |    |    | 3  |    |    |    |
| 10 |    | 3  |    |    |    | 2  | 1  | 0  |    |    |    |    |    |    |    |    |
| 11 |    |    |    |    | 2  |    | 0  | 1  |    |    |    |    |    |    |    |    |
| 12 |    |    |    |    | 1  | 0  |    | 2  |    | 3  |    |    |    |    |    |    |
| 13 |    |    |    |    | 0  | 1  | 2  |    |    |    |    |    |    | 3  |    |    |
| 20 |    |    | 3  |    |    |    |    |    |    | 2  | 1  | 0  |    |    |    |    |
| 21 |    |    |    |    |    |    | 3  |    | 2  |    | 0  | 1  |    |    |    |    |
| 22 |    |    |    |    |    |    |    |    | 1  | 0  |    | 2  |    |    |    |    |
| 23 |    |    |    |    |    |    |    |    | 0  | 1  | 2  |    |    |    | 3  |    |
| 30 |    |    |    | 3  |    |    |    |    |    |    |    |    |    | 2  | 1  | 0  |
| 31 |    |    |    |    |    |    |    | 3  |    |    |    |    | 2  |    | 0  | 1  |
| 32 |    |    |    |    |    |    |    |    |    |    |    | 3  | 1  | 0  |    | 2  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | 0  | 1  | 2  |    |

Adjacency matrix with edge-coloring for  $S_5^2$  in the proof of Lemma 5.

|    | 00 | 01 | 02 | 03 | 04 | 10 | 11 | 12 | 13 | 14 | 20 | 21 | 22 | 23 | 24 | 30 | 31 | 32 | 33 | 34 | 40 | 41 | 42 | 43 | 44 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | x  | x  | x  | x  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | x  |    |    | x  | x  | x  | 3  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 02 | x  | x  |    |    | x  | x  |    |    |    |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 03 | x  | x  | x  |    |    | x  |    |    |    |    |    |    |    |    |    | 4  |    |    |    |    |    |    |    |    |    |
| 04 | x  | x  | x  | x  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2  |    |    |    |    |
| 10 |    | 3  |    |    |    |    |    | x  | x  | x  | x  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 11 |    |    |    |    |    | x  |    |    | x  | x  | x  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 12 |    |    |    |    |    | x  | x  |    |    | x  | x  |    | 4  |    |    |    |    |    |    |    |    |    |    |    |    |
| 13 |    |    |    |    |    | x  | x  | x  |    |    | x  |    |    |    |    |    | 2  |    |    |    |    |    |    |    |    |
| 14 |    |    |    |    |    | x  | x  | x  | x  |    |    |    |    |    |    |    |    |    |    |    |    | 0  |    |    |    |
| 20 |    |    | 1  |    |    |    |    |    |    |    |    | x  | x  | x  | x  |    |    |    |    |    |    |    |    |    |    |
| 21 |    |    |    |    |    |    |    | 4  |    |    | x  |    |    | x  | x  | x  |    |    |    |    |    |    |    |    |    |
| 22 |    |    |    |    |    |    |    |    |    |    | x  | x  |    |    | x  | x  |    |    |    |    |    |    |    |    |    |
| 23 |    |    |    |    |    |    |    |    |    |    | x  | x  | x  |    |    | x  |    | 0  |    |    |    |    |    |    |    |
| 24 |    |    |    |    |    |    |    |    |    |    | x  | x  | x  | x  |    |    |    |    |    |    |    |    | 3  |    |    |
| 30 |    |    |    | 4  |    |    |    |    |    |    |    |    |    |    |    |    | x  | x  | x  | x  |    |    |    |    |    |
| 31 |    |    |    |    |    |    |    |    | 2  |    |    |    |    |    |    | x  |    |    | x  | x  | x  |    |    |    |    |
| 32 |    |    |    |    |    |    |    |    |    |    |    |    | 0  |    |    | x  | x  |    |    | x  | x  |    |    |    |    |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | x  | x  | x  |    |    | x  |    |    |    |    |
| 34 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | x  | x  | x  | x  |    |    |    |    | 1  |    |
| 40 |    |    |    |    | 2  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | x  | x  | x  | x  |
| 41 |    |    |    |    |    |    |    |    |    | 0  |    |    |    |    |    |    |    |    |    |    | x  |    |    | x  | x  |
| 42 |    |    |    |    |    |    |    |    |    |    |    |    |    | 3  |    |    |    |    |    |    |    | x  | x  |    | x  |
| 43 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  |    | x  | x  |    | x  |
| 44 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | x  | x  | x  | x  |

|    | 00 | 01 | 02 | 03 | 04 | 10 | 11 | 12 | 13 | 14 | 20 | 21 | 22 | 23 | 24 | 30 | 31 | 32 | 33 | 34 | 40 | 41 | 42 | 43 | 44 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 |    | 4  | 3  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 4  |    | 2  | 1  | 0  | 3  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 02 | 3  | 2  |    | 0  | 4  |    |    |    |    |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 03 | 2  | 1  | 0  |    | 3  |    |    |    |    |    |    |    |    |    |    | 4  |    |    |    |    |    |    |    |    |    |
| 04 | 1  | 0  | 4  | 3  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2  |    |    |    |    |
| 10 |    | 3  |    |    |    |    | 2  | 1  | 0  | 4  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 11 |    |    |    |    |    | 2  |    | 0  | 4  | 3  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 12 |    |    |    |    | 1  | 0  |    | 3  | 2  |    | 4  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 13 |    |    |    |    | 0  | 4  | 3  |    | 1  |    |    |    |    |    |    |    | 2  |    |    |    |    |    |    |    |    |
| 14 |    |    |    |    | 4  | 3  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    | 0  |    |    |    |
| 20 |    | 1  |    |    |    |    |    |    |    |    | 0  | 4  | 3  | 2  |    |    |    |    |    |    |    |    |    |    |    |
| 21 |    |    |    |    |    |    | 4  |    |    |    | 0  |    | 3  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |
| 22 |    |    |    |    |    |    |    |    |    |    | 4  | 3  |    | 1  | 0  |    |    |    |    |    |    |    |    |    |    |
| 23 |    |    |    |    |    |    |    |    |    |    | 3  | 2  | 1  |    | 4  |    |    | 0  |    |    |    |    |    |    |    |
| 24 |    |    |    |    |    |    |    |    |    |    | 2  | 1  | 0  | 4  |    |    |    |    |    |    |    |    | 3  |    |    |
| 30 |    |    | 4  |    |    |    |    |    |    |    |    |    |    |    |    |    | 3  | 2  | 1  | 0  |    |    |    |    |    |
| 31 |    |    |    |    |    |    |    |    | 2  |    |    |    |    |    |    | 3  |    | 1  | 0  | 4  |    |    |    |    |    |
| 32 |    |    |    |    |    |    |    |    |    |    |    |    | 0  |    |    | 2  | 1  |    | 4  | 3  |    |    |    |    |    |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 0  | 4  |    | 2  |    |    |    |    |    |
| 34 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 0  | 4  | 3  | 2  |    |    |    |    | 1  |    |
| 40 |    |    |    | 2  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 0  | 4  | 3  |
| 41 |    |    |    |    |    |    |    |    |    | 0  |    |    |    |    |    |    |    |    |    |    | 1  |    | 4  | 3  | 2  |
| 42 |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 3  |    |    |    |    |    | 0  | 4  |    | 2  | 1  |
| 43 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 4  | 3  | 2  |    | 0  |
| 44 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 3  | 2  | 1  | 0  |    |

Adjacency matrix with total coloring for  $H_4^2$  in the proof of [Lemma 2'](#).

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 | 0  | 3  | 4  | 2  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 3  | 2  | 0  | 4  |    |    |    |    |    | 5  |    |    |    | 1  |    |    |
| 02 | 4  | 0  | 5  | 1  |    |    | 2  |    |    |    |    |    |    |    | 3  |    |
| 03 | 2  | 4  | 1  | 3  |    |    |    | 0  |    |    |    | 5  |    |    |    |    |
| 10 |    |    |    |    | 0  | 3  | 4  | 2  | 1  |    |    |    | 5  |    |    |    |
| 11 |    |    |    |    | 3  | 1  | 0  | 4  |    |    |    |    |    |    |    |    |
| 12 |    |    | 2  |    | 4  | 0  | 3  | 1  |    |    |    |    |    |    | 5  |    |
| 13 |    |    |    | 0  | 2  | 4  | 1  | 5  |    |    |    | 3  |    |    |    |    |
| 20 |    |    |    |    | 1  |    |    |    | 5  | 3  | 4  | 2  | 0  |    |    |    |
| 21 |    | 5  |    |    |    |    |    |    | 3  | 1  | 0  | 4  |    | 2  |    |    |
| 22 |    |    |    |    |    |    |    |    | 4  | 0  | 2  | 1  |    |    |    |    |
| 23 |    |    |    | 5  |    |    |    | 3  | 2  | 4  | 1  | 0  |    |    |    |    |
| 30 |    |    |    |    | 5  |    |    |    | 0  |    |    |    | 1  | 3  | 4  | 2  |
| 31 |    | 1  |    |    |    |    |    |    |    | 2  |    |    | 3  | 5  | 0  | 4  |
| 32 |    |    | 3  |    |    |    | 5  |    |    |    |    |    | 4  | 0  | 2  | 1  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | 2  | 4  | 1  | 3  |

Adjacency matrix with total coloring for  $H_5^2$  in the proof of Lemma 3'(a).

|    | 00 | 01 | 02 | 03 | 04 | 10 | 11 | 12 | 13 | 14 | 20 | 21 | 22 | 23 | 24 | 30 | 31 | 32 | 33 | 34 | 40 | 41 | 42 | 43 | 44 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 | 0  | 3  | 1  | 4  | 2  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 3  | 1  | 4  | 2  | 0  |    |    |    |    |    | 7  |    |    |    |    |    | 6  |    |    |    |    | 5  |    |    |    |
| 02 | 1  | 4  | 2  | 0  | 3  |    |    | 7  |    |    |    |    |    |    |    |    |    | 6  |    |    |    |    | 5  |    |    |
| 03 | 4  | 2  | 0  | 3  | 1  |    |    |    | 7  |    |    |    |    | 6  |    |    |    |    |    |    |    |    |    | 5  |    |
| 04 | 2  | 0  | 3  | 1  | 4  |    |    |    |    | 7  |    |    |    |    | 6  |    |    |    |    |    | 5  |    |    |    |    |
| 10 |    |    |    |    |    | 1  | 4  | 2  | 0  | 3  | 7  |    |    |    |    | 6  |    |    |    |    |    | 5  |    |    |    |
| 11 |    |    |    |    |    | 4  | 2  | 0  | 3  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 12 |    |    | 7  |    |    | 2  | 0  | 3  | 1  | 4  |    |    |    |    |    |    |    | 5  |    |    |    |    | 6  |    |    |
| 13 |    |    |    | 7  |    | 0  | 3  | 1  | 4  | 2  |    |    |    | 5  |    |    |    |    |    |    |    |    |    | 6  |    |
| 14 |    |    |    |    | 7  | 3  | 1  | 4  | 2  | 0  |    |    |    |    | 5  |    |    |    |    | 6  |    |    |    |    |    |
| 20 |    |    |    |    |    | 7  |    |    |    |    | 2  | 0  | 3  | 1  | 4  | 5  |    |    |    |    | 6  |    |    |    |    |
| 21 |    | 7  |    |    |    |    |    |    |    |    | 0  | 3  | 1  | 4  | 2  |    | 5  |    |    |    |    | 6  |    |    |    |
| 22 |    |    |    |    |    |    |    |    |    |    | 3  | 1  | 4  | 2  | 0  |    |    |    |    |    |    |    |    |    |    |
| 23 |    |    |    | 6  |    |    |    |    | 5  |    | 1  | 4  | 2  | 0  | 3  |    |    |    |    |    |    |    |    | 7  |    |
| 24 |    |    |    |    | 6  |    |    |    |    | 5  | 4  | 2  | 0  | 3  | 1  |    |    |    |    | 7  |    |    |    |    |    |
| 30 |    |    |    |    |    | 6  |    |    |    |    | 5  |    |    |    |    | 3  | 1  | 4  | 2  | 0  | 7  |    |    |    |    |
| 31 |    | 6  |    |    |    |    |    |    |    |    |    | 5  |    |    |    | 1  | 4  | 2  | 0  | 3  |    | 7  |    |    |    |
| 32 |    |    | 6  |    |    |    |    | 5  |    |    |    |    |    |    |    | 4  | 2  | 0  | 3  | 1  |    |    | 7  |    |    |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2  | 0  | 3  | 1  | 4  |    |    |    |    |    |
| 34 |    |    |    |    | 5  |    |    |    |    | 6  |    |    |    |    | 7  | 0  | 3  | 1  | 4  | 2  |    |    |    |    |    |
| 40 |    |    |    |    |    | 5  |    |    |    |    | 6  |    |    |    |    | 7  |    |    |    |    | 4  | 2  | 0  | 3  | 1  |
| 41 |    | 5  |    |    |    |    |    |    |    |    |    | 6  |    |    |    |    | 7  |    |    |    | 2  | 0  | 3  | 1  | 4  |
| 42 |    |    | 5  |    |    |    |    | 6  |    |    |    |    |    |    |    |    |    | 7  |    |    | 0  | 3  | 1  | 4  | 2  |
| 43 |    |    |    | 5  |    |    |    |    | 6  |    |    |    |    |    | 7  |    |    |    |    |    | 3  | 1  | 4  | 2  | 0  |
| 44 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 4  | 2  | 0  | 3  |

Adjacency matrix with total coloring for  $S_5^2$  in the proof of Lemma 4'.

|    | 00 | 01 | 02 | 03 | 04 | 10 | 11 | 12 | 13 | 14 | 20 | 21 | 22 | 23 | 24 | 30 | 31 | 32 | 33 | 34 | 40 | 41 | 42 | 43 | 44 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 | 0  | 3  | 4  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 3  | 1  | 0  | 4  | 2  | 5  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 02 | 4  | 0  | 2  | 1  | 3  |    |    |    |    |    | 5  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 03 | 2  | 4  | 1  | 3  | 0  |    |    |    |    |    |    |    |    |    |    | 5  |    |    |    |    |    |    |    |    |    |
| 04 | 1  | 2  | 3  | 0  | 4  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 5  |    |    |    |    |
| 10 |    | 5  |    |    |    | 0  | 3  | 4  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 11 |    |    |    |    |    | 3  | 1  | 0  | 4  | 2  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 12 |    |    |    |    |    | 4  | 0  | 2  | 1  | 3  |    | 5  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 13 |    |    |    |    |    | 2  | 4  | 1  | 3  | 0  |    |    |    |    |    |    | 5  |    |    |    |    |    |    |    |    |
| 14 |    |    |    |    |    | 1  | 2  | 3  | 0  | 4  |    |    |    |    |    |    |    |    |    |    |    | 5  |    |    |    |
| 20 |    |    | 5  |    |    |    |    |    |    |    | 0  | 3  | 4  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |
| 21 |    |    |    |    |    |    |    | 5  |    |    | 3  | 1  | 0  | 4  | 2  |    |    |    |    |    |    |    |    |    |    |
| 22 |    |    |    |    |    |    |    |    |    |    | 4  | 0  | 2  | 1  | 3  |    |    |    |    |    |    |    |    |    |    |
| 23 |    |    |    |    |    |    |    |    |    |    | 2  | 4  | 1  | 3  | 0  |    |    | 5  |    |    |    |    |    |    |    |
| 24 |    |    |    |    |    |    |    |    |    |    | 1  | 2  | 3  | 0  | 4  |    |    |    |    |    |    |    |    | 5  |    |
| 30 |    |    |    | 5  |    |    |    |    |    |    |    |    |    |    |    | 0  | 3  | 4  | 2  | 1  |    |    |    |    |    |
| 31 |    |    |    |    |    |    |    |    | 5  |    |    |    |    |    |    | 3  | 1  | 0  | 4  | 2  |    |    |    |    |    |
| 32 |    |    |    |    |    |    |    |    |    |    |    |    | 5  |    |    | 4  | 0  | 2  | 1  | 3  |    |    |    |    |    |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2  | 4  | 1  | 3  | 0  |    |    |    |    |    |
| 34 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 2  | 3  | 0  | 4  |    |    |    |    | 5  |
| 40 |    |    |    |    | 5  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 0  | 3  | 4  | 2  | 1  |
| 41 |    |    |    |    |    |    |    |    |    | 5  |    |    |    |    |    |    |    |    |    |    | 3  | 1  | 0  | 4  | 2  |
| 42 |    |    |    |    |    |    |    |    |    |    |    |    | 5  |    |    |    |    |    |    |    | 4  | 0  | 2  | 1  | 3  |
| 43 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 5  |    | 2  | 4  | 1  | 3  | 0  |
| 44 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 2  | 3  | 0  | 4  |

Adjacency matrix with total coloring for  $S_4^2$  in the proof of Lemma 5'.

|    | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 30 | 31 | 32 | 33 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 | 3  | 2  | 1  | 4  |    |    |    |    |    |    |    |    |    |    |    |    |
| 01 | 2  | 4  | 0  | 1  | 3  |    |    |    |    |    |    |    |    |    |    |    |
| 02 | 1  | 0  | 2  | 3  |    |    |    |    | 4  |    |    |    |    |    |    |    |
| 03 | 4  | 1  | 3  | 0  |    |    |    |    |    |    |    |    | 2  |    |    |    |
| 10 |    | 3  |    |    | 1  | 4  | 2  | 0  |    |    |    |    |    |    |    |    |
| 11 |    |    |    |    | 4  | 0  | 3  | 2  |    |    |    |    |    |    |    |    |
| 12 |    |    |    |    | 2  | 3  | 4  | 1  |    | 0  |    |    |    |    |    |    |
| 13 |    |    |    |    | 0  | 2  | 1  | 3  |    |    |    |    |    | 4  |    |    |
| 20 |    |    | 4  |    |    |    |    |    | 0  | 1  | 3  | 2  |    |    |    |    |
| 21 |    |    |    |    |    |    | 0  |    | 1  | 2  | 4  | 3  |    |    |    |    |
| 22 |    |    |    |    |    |    |    |    | 3  | 4  | 1  | 0  |    |    |    |    |
| 23 |    |    |    |    |    |    |    |    | 2  | 3  | 0  | 4  |    |    |    |    |
| 30 |    |    |    | 2  |    |    |    |    |    |    |    |    | 4  | 3  | 0  | 1  |
| 31 |    |    |    |    |    |    |    | 4  |    |    |    |    | 3  | 1  | 2  | 0  |
| 32 |    |    |    |    |    |    |    |    |    |    |    | 1  | 0  | 2  | 3  | 4  |
| 33 |    |    |    |    |    |    |    |    |    |    |    |    | 1  | 0  | 4  | 2  |

## References

- [1] S.N. Afriat, The Ring of Linked Rings, Duckworth, London, 1982.
- [2] D. Arett, S. Dorée, Coloring and counting on the Tower of Hanoi graphs, Math. Mag. 83 (2010) 200–209.
- [3] S. Aumann, A.M. Hinz, C. Petr, Move patterns of the largest disc, in: A. Danek, A.M. Hinz, F. Gauzy (Eds.), Édouard Lucas and His Tower of Hanoi: History, Mathematics, and Cognition, Springer, Berlin (in preparation).
- [4] J.-P. Bode, A.M. Hinz, Results and open questions on the Tower of Hanoi, Congr. Numer. 139 (1999) 113–122.
- [5] T.-H. Chan, A statistical analysis of the Towers of Hanoi problem, Int. J. Comput. Math. 28 (1989) 57–65.
- [6] P. Cull, I. Nelson, Error-correcting codes on the Towers of Hanoi graphs, Discrete Math. 208–209 (1999) 157–175.
- [7] S. Gravier, S. Klavžar, M. Mollard, Codes and  $L(2, 1)$ -labelings in Sierpiński graphs, Taiwanese J. Math. 9 (2005) 671–681.
- [8] A.M. Hinz, The Tower of Hanoi, Enseign. Math. (2) 35 (1989) 289–321.
- [9] A.M. Hinz, The Tower of Hanoi, in: K.P. Shum, E.J. Taft, Z.X. Wan (Eds.), Algebras and Combinatorics, Springer, Singapore, 1999, pp. 277–289.
- [10] A.M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, European J. Combin. 26 (2005) 693–708.
- [11] A.M. Hinz, D. Parisse, On the planarity of Hanoi graphs, Expo. Math. 20 (2002) 263–268.
- [12] A.M. Hinz, D. Parisse, The average eccentricity of Sierpiński graphs, Graphs Combin. (in press).
- [13] A.M. Hinz, A. Schief, The average distance on the Sierpiński gasket, Probab. Theory Related Fields 87 (1990) 129–138.
- [14] W. Imrich, S. Klavžar, D.F. Rall, Topics in Graph Theory, A.K. Peters, Wellesley, MA, 2008.
- [15] M. Jakovac, S. Klavžar, Vertex-, edge-, and total-colorings of Sierpiński-like graphs, Discrete Math. 309 (2009) 1548–1556.
- [16] S. Klavžar, Coloring Sierpiński graphs and Sierpiński gasket graphs, Taiwanese J. Math. 12 (2008) 513–522.
- [17] S. Klavžar, U. Milutinović, Graphs  $S(n, k)$  and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47 (122) (1997) 95–104.
- [18] S. Klavžar, U. Milutinović, C. Petr, 1-perfect codes in Sierpiński graphs, Bull. Austral. Math. Soc. 66 (2002) 369–384.
- [19] S. Klavžar, B. Mohar, Crossing numbers of Sierpiński-like graphs, J. Graph Theory 50 (2005) 186–198.
- [20] R.E. Korf, Linear-time disk-based implicit graph search, J. ACM 55 (2008) Art. 26.
- [21] D. Parisse, On some metric properties of the Sierpiński graphs  $S(n, k)$ , Ars Combin. 90 (2009) 145–160.
- [22] D. Romik, Shortest paths in the Tower of Hanoi graph and finite automata, SIAM J. Discrete Math. 20 (2006) 610–622.
- [23] R.S. Scorer, P.M. Grundy, C.A.B. Smith, Some binary games, Math. Gaz. 28 (280) (1944) 96–103.
- [24] A. Soifer, The Mathematical Coloring Book, Springer, New York, 2009.
- [25] Q. Stierstorfer, Codes on tower graphs, in: A. Danek, A.M. Hinz, F. Gauzy (Eds.), Édouard Lucas and His Tower of Hanoi: History, Mathematics, and Cognition, Springer, Berlin (in preparation).
- [26] P.K. Stockmeyer, at the Workshop on the Tower of Hanoi and Related Problems, Maribor, 2005.
- [27] D.B. West, Introduction to Graph Theory, second ed., Prentice Hall, Upper Saddle River, NJ, 2001.
- [28] R.J. Wilson, Introduction to Graph Theory, third ed., Longman, Harlow, 1985.
- [29] H.P. Yap, Total Colourings of Graphs, Springer, Berlin, 1996.